07/08/23

UNIT - 5

VECTOR INTEGRATION :-

Work Done : (by a force) : If $\bar{F}$ represents the force, vector acting acting on a partical moving along the arc. Then the work done during a small displacement is $\bar{F} \cdot d\bar{r}$.

Hence the work done $\bar{F}$ during the displacement A to B is given by the line integral $\int_A^B \bar{F} \cdot d\bar{r}$

1) Evaluate $\int_C \bar{F} \cdot d\bar{r}$. Where, $\bar{F} = x^2\bar{i} + y^2\bar{j}$ and c is the curve $y = x^2$ in xy plane from $(0,0)$ to $(1,1)$

Solution:- Given: $\bar{F} = x^2\bar{i} + y^2\bar{j}$

$\bar{F}$ along the curve $y = x^2$

Let $\bar{r} = x\bar{i} + y\bar{j}$

$\Rightarrow d\bar{r} = dx\bar{i} + dy\bar{j}$

Now, $\bar{F} \cdot d\bar{r} = (x^2\bar{i} + y^2\bar{j}) \cdot (dx\bar{i} + dy\bar{j})$

$\bar{F} \cdot d\bar{r} = x^2 dx + y^2 dy$

Since, $y = x^2$

$\Rightarrow dy = 2x\,dx$

$\bar{F} \cdot d\bar{r} = x^2 dx + x^4\, 2x\, dx$

$\bar{F} \cdot d\bar{r} = x^2 dx + 2x^5 dx$

Here, $\int_C \bar{F} \cdot d\bar{r} = \int_0^1 (x^2 dx + 2x^5 dx)$ for $(0,0)$ to $(1,1)$

$= \left[\frac{x^3}{3}\right]_0^1 + 2\left[\frac{x^6}{6}\right]_0^1$

$= \frac{1}{3} + \frac{2}{6} = \frac{2}{3}$

$\therefore \int_C \bar{F} \cdot d\bar{r} = \int_0^1 x^2 dx + 2x^5 dx = \frac{2}{3}$

$$\int_C \bar{F} \cdot d\bar{r} = \int_C x^2 dx + y^2 dy$$

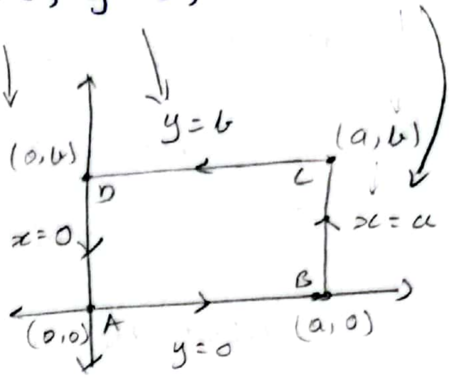
$$= \int_C x^2 dx + \int_C y^2 dy$$

$$= \int_0^1 x^2 dx + \int_0^1 y^2 dy$$

$$= \left(\frac{x^3}{3}\right)_0^1 + \left(\frac{y^3}{3}\right)_0^1$$

$$= 1/3 + 1/3 = 2/3$$

2Q) If $\bar F = (x^2+y^2)\,\bar i - (2xy)\,\bar j$. Evaluate $\int_C \bar F\cdot d\bar r$ where c is rectangle in xy plane. Is bounded by $y=0,\ y=b;\ x=0,\ x=a$.



Solution: Given: $\bar F\ (x^2+y^2)\bar i - (2xy)\,\bar j$

$$\bar r = x\bar i + y\bar j + 0\bar k$$

$$d\bar r = dx\,\bar i + dy\,\bar j.$$

Hence,

$$\bar F\cdot d\bar r = \left[(x^2+y^2)\,\bar i - (2xy)\,\bar j\right]\cdot(dx\, i + dy\, j)$$

$$\bar F\cdot d\bar r = (x^2+y^2)\,dx - 2xy\,dy$$

Here,

$$\int_C \bar F\cdot d\bar r = \underset{\substack{(y=0)\\ x=0}}{\overset{x=a\ (AB)}{\int}} \bar F\cdot d\bar r + \underset{\substack{(x=a)\\ y=0}}{\overset{y=b\ (BC)}{\int}} \bar F\cdot d\bar r + \underset{\substack{(y=b)\\ x=a}}{\overset{x=0\ (CD)}{\int}} \bar F\cdot d\bar r + \underset{\substack{(x=0)\\ y=b}}{\overset{y=0\ (DA)}{\int}} \bar F\cdot d\bar r$$

$$= \int_0^a x^2\,dx + \int_0^b -2ay\,dy + \int_a^0 (x^2+b^2)\,dx + \int_b^0 -2(0)\,y\,dy$$

$$= \cancel{\left[\frac{x^3}{3}\right]_0^a} + -2a\left[\frac{y^2}{2}\right]_0^b + \left[\cancel{\left[\frac{x^3}{3}\right]_a^0} + b^2\left[x\right]_a^0\right] + 0$$

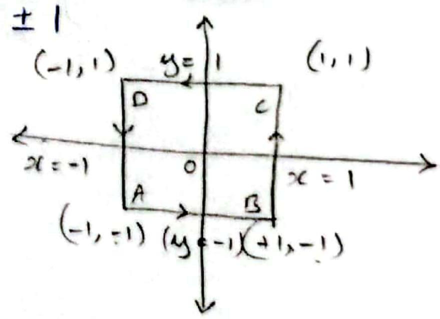
$$= -a\left[b^2-0\right] + b^2\left[0-a\right]$$

$$= -ab^2 \mp ab^2$$

$$= \cancel{0}\ - 2ab^2.$$

1Q) E

3Q) Evaluate the line integral $\int_C (x^2+xy)\,dx + (x^2+y^2)\,dx$ where c is the square formed by lines $x=\pm 1,\ y=\pm 1$



09/08/23 M2

Surface Integral :-

Let $\bar{F} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ are continuous and differentiable function of x, y, z other than and let 's' be the region of the surface. Divided the region into 'm' sub regions of area $ds_1, ds_2, ds_3, \ldots ds_m$ and $\bar{n}$ is the unit normal to ds then

$$\int_S \bar{F}.\bar{n}\, ds$$

Note: • Let 'R_1' be the projection of 's' on x, y plane then

$$\int_S \bar{F}.\bar{n}\, ds = \iint_R \frac{\bar{F}.\bar{n}}{|\bar{n}.\bar{k}|} \cancel{dy\,dz}\; dx\, dy.$$

• Let 'R_1' is the projection of 's' on yz plane

$$\int_S \bar{F}\,\bar{n}\, ds = \iint_R \frac{\bar{F}.\bar{n}}{|\bar{n}.\bar{i}|} dz\, dy$$

• Let R_1 be the projection of 's' on xz plane

$$\int_S \bar{F}\,\bar{n}\, ds = \iint_R \frac{\bar{F}.\bar{n}}{|\bar{n}.\bar{j}|} dz\, dx$$

1Q) Evaluate $\int_S \bar{F}.\bar{n}\, ds$ where $\bar{F} = z\bar{i} + x\bar{j} \cancel{+ 3y\bar{k}} - 3y^2 z\bar{k}$ is the surface $x^2+y^2=16$ included in first octate between $z=0$ and $z=5$

Given:- $\bar{F} = z\bar{i} + x\bar{j} - 3y^2 z\bar{k}$

Surface $x^2+y^2=16. \Rightarrow \phi$

$$\Rightarrow \bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2xi + 2yj}{2\,\cancel{\sqrt{8}}\sqrt{x^2+y^2}} = \frac{xi+yj}{4}$$

Given it is first octant between $z=0, z=5$.

$$\int_S \bar{F}.\bar{n}\, ds = \iint_R \frac{\bar{F}.\bar{n}}{|\bar{n}.\bar{j}|}\, dy\, dz \quad (yz \text{ Plane}) \quad (x=0)$$

$$= \int_{y=0}^{4}\int_{z=0}^{5} \frac{(z\bar{i}+x\bar{j}-3y^2z\bar{k})\left(\frac{x\bar{i}+y\bar{j}}{4}\right)}{\left|\left(\frac{x i+y j}{4}\right).\, i\right|}\, dy\, dz$$

$z=0, z=5$ (Given)

Surface $\Rightarrow x^2+y^2=16$

$\because x=0$

$\Rightarrow y^2=16$

$y=\pm 4$

$y=0,\ y=4$ (1st. octant)

$$= \int_{y=0}^{4}\int_{z=0}^{5} \frac{\left(\frac{xz}{4}+\frac{xy}{4}\right)}{\left(\frac{x}{4}\right)}\, dy\, dz = \iint \frac{x/4\,(z+y)}{x/4}\, dy\, dz$$

$$= \int_{y=0}^{4}\int_{z=0}^{5} [z+y]\, dz\, dy$$

$$= \int_{y=0}^{4} \left[\frac{z^2}{2}+yz\right]_0^5 dy = \int_{y=0}^{4}\left[\frac{25}{2}+5y-0\right] dy$$

$$= \left[\frac{25}{2}y+5\frac{y^2}{2}\right]_0^4$$

$$= \frac{25}{2}(4)+\frac{5(16)}{2}-[0+0]$$

$$= 50+40$$

$$= 90.\ \text{unit}^2.$$

2) Evaluate $\iint_S \bar{F}.\bar{n}\, ds$ where $\bar{F} = 12x^2y\,\bar{i} - 3yz\,\bar{j} + 2z\bar{k}$ and 'S' is ~~the~~ portion of plane $x+y+z=1$ in 1st octant.

$(x+y=1)$

$x=1-y \quad x=0$

$y=1 \quad y=0$

Given :- $\bar{F} = 12x^2y\,\hat{i} - 3yz\,\hat{j} + 2z\,\bar{k}$

Surface $\Rightarrow x + y + z = 1 \Rightarrow \phi = x + y + z - 1$

$$\Rightarrow \bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\bar{i} + \bar{j} + \bar{k}}{\sqrt{1+1+1}} = \frac{\bar{i} + \bar{j} + \bar{k}}{\sqrt{3}}$$

Let it be xy plane $\Rightarrow z = 0$.

$$\Rightarrow x + y + z = 1$$
$$x + y = 1$$
$$\Rightarrow x = 1 - y$$

Limits for $x \Rightarrow (0, 1-y)$

$1 - y = 0 \Rightarrow y = 1$

Limits for $y \Rightarrow (0, 1)$

for xy plane, $\int_S \bar{F}\cdot\bar{n}\,ds = \iint_R \dfrac{\bar{F}\cdot\bar{n}}{|\bar{n}\cdot\bar{k}|}\,dx\,dy$

$$= \int_{y=0}^{1}\int_{x=0}^{1-y} \frac{(12x^2y\,\hat{i} - 3yz\,\hat{j} + 2z\,\hat{k})\cdot\left(\frac{\bar{i}+\bar{j}+\bar{k}}{\sqrt{3}}\right)}{\left(\frac{\bar{i}+\bar{j}+\bar{k}}{\sqrt{3}}\right)(\bar{k})}\,dx\,dy$$

$$= \int_{y=0}^{1}\int_{x=0}^{1-y} \frac{\frac{1}{\sqrt{3}}\left[12x^2y - 3yz + 2z\right]}{1/\sqrt{3}}\,dx\,dy$$

$$= \int_{y=0}^{1} \left(12y\left[\frac{x^3}{3}\right]_0^{1-y} - 3yz\,[x]_0^{1-y} + 2z\,[x]_0^{1-y}\right) dy$$

$\because z = 0,$

$$= \int_{y=0}^{1} \left(12y\left[\frac{x^3}{3}\right]_0^{1-y} - 0 + 0\right) dy = \int_{y=0} \frac{12}{3}y(1-y)^3\,dy$$

$$= 4\int_{y=0}^{1} y\left[1 - y^3 - 3y + 3y^2\right] dy = 4\left[\frac{y^2}{2} - \frac{y^5}{5} - \frac{3y^3}{3} + \frac{3y^4}{4}\right]_0^1 = 4\left(\frac{1}{4}\right)$$

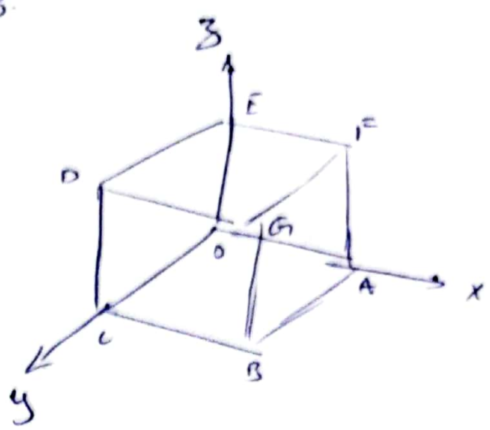
$= \cancel{13}.\cancel{2}\ \cancel{2}\cdot 1/5$ $= \cancel{4}(1/20)$

10/07/2023

Evaluate $\iint_S F.\bar{n}\, ds$ if $\bar{F} = 2xy\bar{i} + yz^2\bar{j} + xz\bar{k}$ over

parallapiped $x=0, y=0, z=0, x=2, y=1, z=3$

Given: $\bar{F} = 2xy\bar{i} + yz^2\bar{j} + xz\bar{k}$



a) In xy plane:

i) OABC xy plane $z=0 \Rightarrow \bar{n} = -\bar{k}$

$$\int_S F.\bar{n}\, ds = \iint_S \frac{\bar{F}\cdot\bar{n}}{|\bar{n}\cdot\bar{k}|}\, ds$$

$$= \int_{y=0}^{1}\int_{x=0}^{2} \frac{(2xy\bar{i} + yz^2\bar{j} + xz\bar{k})(-\bar{k})}{|(-\bar{k}\cdot\bar{k})|}\, dx\, dy$$

$\therefore z=0$

$= 0$ unit^2.

ii) DEFG $z=3$

$$= \int_{y=0}^{1}\int_{x=0}^{2} \frac{(2xy\bar{i} + yz^2\bar{j} + xz\bar{k})(-\bar{k})}{|(-\bar{k})\cdot(\bar{k})|}\, dx\, dy$$

$\therefore z=3$

$$\int_{y=0}^{1}\int_{x=0}^{2} \frac{-3x}{-1}\, dx\, dy = \int_{y=0}^{1} 3\left[\frac{x^2}{2}\right]_0^2 dy$$

$$= 6\int_{y=0}^{1} dy = 6[x]_0^1$$

$= 6$ unit^2

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b) in yz plane:

i) OCDE yz plane $x=0 \Rightarrow \bar{n} = -\bar{i}$

$$\int_S \bar{F}.\bar{n}\, ds = \iint_S \frac{\bar{F}.\bar{n}}{|\bar{n}.\bar{i}|}\, ds$$

$\cancel{\because x=0}$

$$= \int_{z=0}^{3}\int_{y=0}^{1} \frac{(2xy\,\bar{i} + yz^2\,\bar{j} + xz\,\bar{k})(-\bar{k})}{|-\bar{i}\cdot\bar{i}|}\, dy\, dz$$

$\because x=0$

$$= \cancel{\int_{z=0}^{3}\int_{y=0}^{1} \frac{-2xy}{-1}\, dy\, dz = \int +2x\left[\frac{y^2}{2}\right]_0^1 dz}$$

$$= \iint 0\, dy dz$$

$= 0$ unit^2

ii) ABGF $x=3$

$$= \int_{z=0}^{3}\int_{y=0}^{1} \frac{-2xy}{-1}\, dy\, dz$$

$$= \int_{z=0}^{3}\int_{y=0}^{1} 6y\, dy\, dz = \int_{z=0}^{3} 6\left[\frac{y^2}{2}\right]_0^1 dz$$

$$= 3\left[z\right]_0^3$$

$= 9$ unit^2

c) in xz plane:

xz $y=0 \Rightarrow \bar{n} = -\bar{j}$

i) $\Rightarrow$ for $y=0$ OAFE

$$\int_S \bar{F}.\bar{n}\, ds = 0$$

ii) for $y=1$ BCDG

$$\int_S \bar{F}.\bar{n}\, ds = \iint_{x\ z} \frac{(yz^2\bar{j})(-\bar{j})}{|-\bar{j}\cdot\bar{j}|}\, dz\, dx$$

$$= \int_{x=0}^{2} \left[\frac{z^3}{3}\right]_0^3 dx = 9\left[x\right]_0^2 = 18 \text{ unit}^2$$

Total surface area $= 0+6+9+18$
$= 30$ unit^2.

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* Volume Integrals:

Let $\bar F = f_1\bar i + f_2\bar j + f_3\bar k$ are the functions of x, y, z and also $dv = dx\,dy\,dz$ then, Volume Integral is

$$\int_V \bar F .\, dv = \iiint_{z\,y\,x} (F_1\bar i + F_2\bar j + F_3\bar k)\,dx\,dy\,dz$$

Q) If $\bar F = 2xz\bar i \overset{-x\bar j}{\cancel{-z\bar k}} + y^2\bar k$ Evaluate $\int_V \bar F\,dv$ where V is the region bounded by $x=0,\ x=2,\ y=0;\ y=6,\ z=x^2,\ z=4.$

Given: $\bar F = 2xz\bar i - x\bar j + y^2\bar k$

$$\because \int_V \bar F\,dv = \iiint_{z\,y\,x} (F_1\bar i + F_2\bar j + F_3\bar k)\,dx\,dy\,dz$$

$$= \int_{z=x^2}^{4}\int_{y=0}^{y=6}\int_{x=0}^{x=2} 2xz\bar i - x\bar j + y^2\bar k\ dx\,dy\,dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} 2xz\bar i - x\bar j + y^2\bar k\ dz\,dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{6} 2x\left[\frac{z^2}{2}\right]_{x^2}^{4} - x\,[z]_{x^2}^{4} + y^2[z]_{x^2}^{4}\ dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{6} \left[x(16) - x(x^2)\right]\bar i - \left[x(4) - x(x^2)\right]\bar j + \left[y^2(4) - y^2(x^2)\right]\bar k\ dy\,dx$$

$$= \int_{x=0}^{2} (16x - x^3)\,[y]_0^6\,\bar i - \left[4x - x^3\right][y]_0^6\,\bar j + (4 - x^2)\left[\frac{y^3}{3}\right]_0^6 dx$$

$$= 96\,[x]_0^2 - 6\left[\frac{x^4}{4}\right]_0^2 \bar i - 24\left[\frac{x^2}{2}\right]_0^2 + 6\left[\frac{x^4}{4}\right]_0^2 \bar j + 228\,[x]_0^2 - 72\left[\frac{x^3}{3}\right]\bar k$$

$$= 168\bar i - 24\bar j + 648\bar k$$

$(128\bar i - 24\bar j + 384\bar k)$

Greens Theorom in a plane: (Transformation between line integral and double integral).

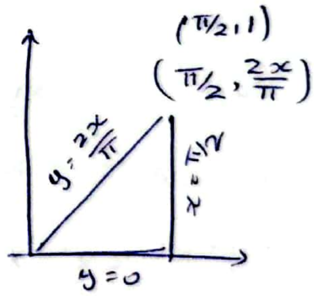
If "R" is clossed region in xy plane, bounded by a simple closed curve "C" and if "M" and "N" are continous function of x and y having continous derivative in R.

Then, $$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

① Evaluate by Green's Theorom $\oint_C (y - \sin x)\,dx + \cos x\,dy$ where c is the Δ enclosed by lines $y=0$, $x = \pi/2$, $\pi y = 2x$

Given:-

$$\oint_C (y - \sin x)\,dx + \cos x\,dy$$



$$\because \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

$\Rightarrow M = y - \sin x$ $\qquad$ $N = \cos x$

$\frac{\partial M}{\partial y} = 1 - \cancel{\cos} 0$ $\qquad$ $\frac{\partial N}{\partial x} = -\sin x$

$$= \iint_R (-\sin x - 1)\,dx\,dy$$

$$\Rightarrow \int_0^1 \int_0^{\pi/2} (-\sin x - 1)\,dx\,dy$$

$$= -1 \int_0^1 \left[-\cos x + x \right]_0^{\pi/2} dy$$

$$= -1 \int_0^1 \left(-\cos \pi/2 \overset{+\cos 0^\circ}{+} \pi/2 \right) dy = \left(+\cos \pi/2 - \cos 0 - \pi/2 \right) \left[y \right]_0^1$$

$$= -1 - \pi/2$$

$$\int_0^{\pi/2}\int_0^{2x/\pi} (-\sin x - 1)\, dy\, dx \qquad \int uv = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots$$

$$\int_0^{\pi/2} (-\sin x - 1)\, [y]_0^{2x/\pi}\, dx$$

$$\int_0^{\pi/2} \frac{2x}{\pi}(-\sin x - 1)\, dx = \int_0^{\pi/2} -\frac{2x}{\pi}\sin x - \frac{2x}{\pi}\, dx$$

$$= -\frac{2}{\pi}\Big[x(-\cos x) - (1)(-\sin x)\Big]_0^{\pi/2} - \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi/2}$$

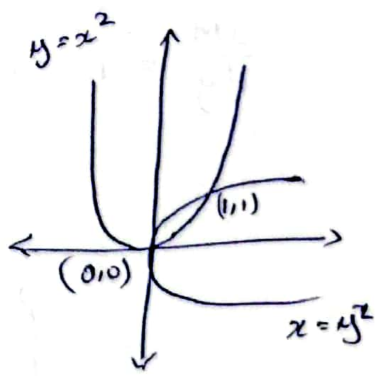
$$= -\frac{2}{\pi}\Big[\tfrac{\pi}{2}(-\cos\tfrac{\pi}{2}) + \sin\tfrac{\pi}{2} - \sin 0\Big] - \frac{2}{\pi}\left[\frac{(\pi/2)^2}{2}\right]$$

$$= -\frac{2}{\pi}\Big[0 + 1 - 0\Big] - \frac{2}{\pi}\left[\frac{\pi^2}{8}\right] \qquad 3.14$$

$$-\frac{2}{\pi} - \frac{\pi}{4} = -1.422 \text{ unit}^2.$$

Q: Using Green's theorem evaluate $\oint_C (2xy - x^2)\,dx + (x^2+y^2)\,dy$ where 'c' is the closed curve in xy plane bounded by the curves $y = x^2$ & $y^2 = x$.

Given: $\oint_C (2xy - x^2)\,dx + (x^2+y^2)\,dy$



$M = 2xy - x^2 \qquad N = x^2+y^2$

$\frac{dM}{dy} = 2x \qquad \frac{dN}{dx} = 2x$

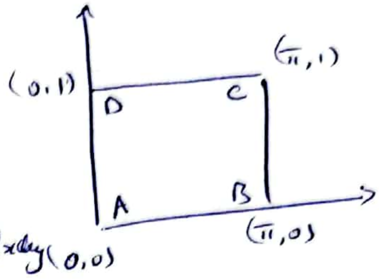
$$\oint_C (2xy - x^2)\,dx + (x^2+y^2)\,dy = \iint_R 2x - 2x\; dx\,dy$$

$$= \int_{y=0}^{1}\int_{x=\sqrt{y}}^{x=y^2} (2x - 2x)\, dx\, dy = 0$$

Evaluate by Green's Theorem $\oint_C (x^2 - \cosh y)\,dx + (y + \sin x)\,dy$

where c is rectangles $(0,0), (\pi,0), (\pi,1), (0,1)$

Given: $\oint_C (x^2 - \cosh y)\,dx + (y + \sin x)\,dy$



$$\therefore \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Here, $M = x^2 - \cosh y$ $\quad N = y + \sin x$

$$\frac{\partial M}{\partial y} = h \sin hy \qquad \frac{\partial N}{\partial x} = \cos x$$

$$\oint_C (x^2 - \cosh y)\,dx + (y + \sin x)\,dy = \int_{y=0}^{1}\int_{x=0}^{\pi} (\cos x - h \sin hy)\,dx\,dy$$

$$= \int_{y=0}^{1} [\sin x]_0^{\pi} - h \sin hy\,[x]_0^{\pi}\,dy$$

$$= \int_0^1 [\sin \pi - h\pi(\sin hy)]\,dy$$

$$= -h\pi \int_0^1 \sin hy\,dy$$

$$= \overset{-1}{\cancel{-h}}\pi \left[\frac{-\cos hy}{\cancel{h}}\right]_0^1$$

$$= -\pi\left[-\cosh - \cosh(0)\right]$$

$$= \pi \cosh - \pi$$

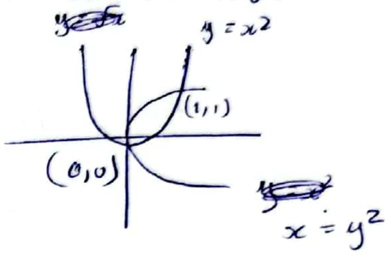
~~Using Green's Theorem, evaluate $\int_C (2xy - x^2)dx + (x^2+y^2)dy$. where c is closed curve in xy plane bounded by $y=x^2$~~

14/08/23

Verify Green's Theorem in the plane for

$\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where c is bounded

by $y = \sqrt{x}$, $y = x^2$



Sol^n $M = 3x^2 - 8y^2$ $N = 4y - 6xy$

$\frac{\partial M}{\partial y} = -16y$ $\frac{\partial N}{\partial x} = -6y$

$$\because \oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\oint_C (3x^2-8y^2)dx + (4y-6xy)dy = \iint_R (-6y+16y)\,dx\,dy$$

LHS

$$\Rightarrow \int_C (3x^2-8y^2)dx + (4y-6xy)dy + \int_C (3x^2-8y^2)dx + (4y-6xy)dy$$

$C \Rightarrow y = x^2$ $C \Rightarrow x = y^2$

$\Rightarrow dy = 2x\,dx$ $dx = 2y\,dy$

$$= \int_{x=0}^{1} (3x^2 - 8x^4)dx + (4x^2 - 6x^3)2x\,dx \quad \int_{y=1}^{0} \left((3y^4 - 8y^2)2y + 4y - 6y^3\right)dy$$

$$= \int_0^1 (3x^2 + 8x^3 - 20x^4)\,dx + \int_1^0 6y^5 - 22y^3 + 4y\,dy$$

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$$= \left[\cancel{3}^{1}\left(\frac{x^3}{\cancel{3}}\right) + \cancel{8}^{2}\left(\frac{x^4}{\cancel{4}}\right) - \cancel{20}^{-4}\left(\frac{x^5}{\cancel{5}}\right)\right]_0^1 + \left[\cancel{6}^{1}\left(\frac{y^6}{\cancel{6}}\right) - \cancel{22}^{-11}\left(\frac{y^4}{\cancel{4}_2}\right) + \cancel{4}^{2}\left(\frac{y^2}{\cancel{2}}\right)\right]_1^0$$

$$= \left[x^3 + 2x^4 - 4x^5\right]_0^1 + \left[y^6 - \frac{11}{2}y^4 + 2y^2\right]_1^0$$

$$= 1 + 2(1) - 4(1) - 0 + 0 - \left[1 - \frac{11}{2}(1) + 2(1)\right]$$

$$= -1 - \left(3 - \frac{11}{2}\right)$$

$$= -1 - \left(\frac{6-11}{2}\right)$$

$$= \frac{5}{2} - 1 = \frac{3}{2}.$$

$LHS = \frac{3}{2}.$

RHS

$$\int_{x=0}^{x=1}\int_{y=x^2}^{y=\sqrt{x}} 10y \, dy \, dx = \int_0^1 \cancel{10}^{5}\left[\frac{y^2}{\cancel{2}}\right]_{x^2}^{\sqrt{x}} dx$$

$$= 5\int_0^1 (x - x^4) \, dx$$

$$= 5\left[\left(\frac{x^2}{2}\right)_0^1 - \left(\frac{x^5}{5}\right)_0^1\right]$$

$$= 5\left[\frac{1}{2} - \frac{1}{5}\right]$$

$$= \cancel{5\left(\frac{3-1}{6}\right)} \; \cancel{5\left(\frac{2}{6}\right)} \; 5\left[\frac{5-2}{\cancel{10}_2}\right]$$

$$= \cancel{\frac{5}{3}} = \frac{3}{2}$$

$RHS = 3/2$

$LHS = RHS$

Hence Verified.

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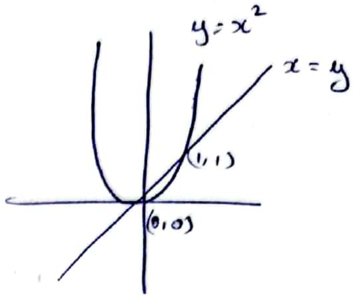
Q) Verify Green's Theorom for $\oint_C (xy+y^2)dx + x^2 dy$ where c is bounded by $y=x$, $y=x^2$



$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

$M = (xy+y^2)$ $\qquad N = x^2$

$\frac{\partial M}{\partial y} = x + 2y$ $\qquad \frac{\partial N}{\partial x} = 2x$

$$\Rightarrow \oint_C (xy+y^2)dx + x^2 dy = \int_{x=0}^{x=1} \int_{y=x^2}^{y=x} 2x - (x+2y)\, dx\, dy$$

LHS

$$\oint_C (xy+y^2)dx + x^2 dy = \int_0^1 \left(x(x^2) + (x^2)^2\right) dx + x^2 (2x) dx \quad c \Rightarrow y = x^2$$

$\boxed{\begin{array}{l} y = x^2 \\ dy = 2x\, dx \end{array}}$

$$+ \int_1^0 (y(y) + y^2) dy + y^2 dy \quad c \Rightarrow x = y.$$

$\boxed{\begin{array}{l} x = y \\ dx = dy \end{array}}$

$$= \int_0^1 (x^3 + x^4 + 2x^3) dx + \int_1^0 (y^2 + y^2 + y^2) dy$$

$$= \int_0^1 (3x^3 + x^4) dx + \int_1^0 3y^2 dy$$

$$= \left[3\frac{x^4}{4} + \frac{x^5}{5} \right]_0^1 + \left[\cancel{3} \frac{y^3}{\cancel{3}} \right]_1^0$$

$$= \frac{3}{4} + \frac{1}{5} - 1$$

$$\frac{15+4-20}{20} = \frac{-1}{20}.$$

$$RHS = \int_{x=0}^{1} \int_{y=x^2}^{y=x} x - 2y \; dx\,dy$$

$$= \int_0^1 \int_{x^2}^{x} (x-2y)\,dy\,dx = \int_0^1 x\,(y)_{x^2}^{x} - \cancel{2}\left[\frac{y^2}{\cancel{2}}\right]_{x^2}^{x} dx$$

$$= \int_0^1 x\,[x - x^2] - [x^2 - x^4]\; dx$$

$$= \int_0^1 \cancel{x^2} - x^3 - \cancel{x^2} + x^4 \; dx$$

$$= \left[\frac{-x^4}{4} + \frac{x^5}{5}\right]_0^1$$

$$= \frac{-1}{4} + \frac{1}{5}$$

$$RHS = \frac{-1}{20}$$

LHS = RHS

Hence Verified.

H.W ① Verify Green's Theorem in plane $\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$

c is square $(0,0)\ (2,0)\ (2,2)\ (0,2)$.

② Evaluate $\oint_C (2x^2 - y^2)\,dx + (x^2+y^2)\,dy$ by green's Theorem.

C is xy plane Semicircle $x^2+y^2 = a^2$.

16/08/2023.

Stokes Theorem:- (Transformation between line and Surface)

Let 'S' be a open surface bounded by a closed non intersecting curve Curve "C". If $\bar{f}$ is any diff. vector point function

Then, $\int_C \bar{F} \cdot d\bar{r} = \int_C \text{curl } \bar{F} \cdot \bar{n}\, ds$

Note:- i) On xy plane, $\int_S \bar{k} \cdot \bar{n}\, ds = \iint dx\,dy$

on yz plane, $\int_S \bar{i} \cdot \bar{n}\, ds = \iint dy\,dz$

on zx plane, $\int_S \bar{j} \cdot \bar{n}\, ds = \iint dz\,dx$

① Verify Stokes Theorem $\bar{F} = (2x-y)i - yz^2 \cdot j - y^2 z\, \bar{k}$ over the upper half of sphere $x^2+y^2+z^2 = 1$ bounded by projection of xy plane. $\Rightarrow x^2+y^2 = 1.\ (z=0)$
$(z=0)$

$\because$

$\int_S \bar{k} \cdot \bar{n}\, ds = \iint dx\,dy$ $\qquad d\bar{r} = x\bar{i} + y\bar{j}$.

$\int_C \bar{F} \cdot d\bar{r} = \int_C \text{Curl } \bar{F} \cdot \bar{n}\, ds$.

LHS $\int_C \bar{F} \cdot d\bar{r} = \int_C (2x-y)\,dx + (-yz^2)\,dy$

curve $\Rightarrow x^2+y^2=1$

$\Rightarrow x = \cos\theta,\ y = +\sin\theta$

$dx = -\sin\theta\, d\theta,\ dy = \cos\theta\, d\theta$

$$\Rightarrow \int_C \bar{F}\cdot d\bar{r} = \int^{2\pi} (2\cos\theta - \sin\theta)\cdot \sin\theta\, d\theta + -\sin\theta\,(0)\cos\theta\, d\theta.$$

$$= \int_0^{2\pi} (-2\sin\theta\cos\theta + \sin^2\theta)\, d\theta$$

$$= \int_0^{2\pi} \left[-\sin 2\theta + \frac{1-\cos 2\theta}{2}\right] d\theta$$

$$= \left[\frac{+\cos 2\theta}{2}\right]_0^{2\pi} + \frac{1}{2}[\theta]_0^{2\pi} - \frac{1}{2}\left[\frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2} + \frac{2\pi}{2} - \frac{1}{2}\left[\frac{0}{2}\right] - \frac{1}{2}$$

$= \pi$

RHS $\Rightarrow \int_S \text{curl } F\cdot\bar{n}\, ds = \int_S \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}\cdot\bar{n}\, ds$

$$= \int_S [-z^2 + y2z]\,\bar{i} - [0-0]\,j + [0+1]\,k$$

$z=0$

$$= \int_S \bar{k}\cdot\bar{n}\, ds$$

$$= \iint dx\, dy \quad [\because \text{circle } (x,y) \to (r,\theta)]$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r\, dr\, d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^1 d\theta = \int_0^{2\pi}\left[\frac{1}{2}\right] d\theta = \left[\frac{\theta}{2}\right]_0^{2\pi}$$

$$= \frac{2\pi}{2} = \pi$$

LHS = RHS

Hence Proved

(2) Verify Stokes Theorem $\bar{F} = -y^3\bar{i} + x^3\bar{j}$ where S is circular disk $x^2 + y^2 \leq 1$, $z = 0$.

(3) Verify Stokes Theorem $\bar{F} = (x^2+y^2)\bar{i} - 2xy\,\bar{j}$ taken around rectangle bounded by lines $x = \pm a$, $y = 0$, $y = b$

$$\bar{F} = (x^2+y^2)\bar{i} - 2xy\,\bar{j}$$

LHS

$$\int_C \bar{F}\cdot d\bar{r} = \int (x^2+y^2)\,dx + (-2xy)\,dy$$

$y=0$: $(-a,0) \to (a,0)$ $\qquad$ $x = a$, $dx = 0$

$$= \int_{-a}^{a} x^2\,dx + \int_{y=0}^{b} (a^2+y^2)\,0 + (-2ya)\,dy$$

$y = b$, $dy = 0$ $\qquad$ $x = -a$

$$+ \int_{a}^{-a} (x^2+b^2)\,dx + 0 + \int_{b}^{0} (-a^2+y^2)\,0 + -2(-a)y\,dy$$

$$= \left[\frac{x^3}{3}\right]_{-a}^{a} + -2a\left[\frac{y^2}{2}\right]_0^b + \left[\frac{x^3}{3}\right]_a^{-a} + b^2\left[x\right]_a^{-a}$$

$$= \cancel{\frac{a^3}{3}} + \cancel{\frac{a^3}{3}} - \cancel{\frac{2ab^2}{2}} - \cancel{\frac{a^3}{3}} - \cancel{\frac{a^3}{3}} + b^2\left[-2a\right] + 2a\left[\frac{y^2}{2}\right]_b^0 + \frac{2a\left[-b^2\right]}{2}$$

$$= -ab^2 - ab^2 - 2ab^2$$

$$= -4ab^2$$

$$= \int_C \text{curl}\,F\cdot\bar{n}\,ds = \iint \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -xy & 0 \end{vmatrix}$$

$$= \bar{k}\left[-y - 2y\right]\bar{n}\,ds$$

$=$

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$$i\left[\frac{d}{dy}(0) - \frac{d}{dz}(-xy)\right] - j\left[\frac{d}{dx}(0) - \frac{d}{dz}\left[x^2+y^2\right]\right] + k\left[\frac{d}{dx}(-xy) - \frac{d}{dy}\left[x^2+y^2\right]\right]$$

$$= \int i\,[0] - j\,[0] + k\left[-y - 2y\right] \hat{n}\; ds$$

$$= \int \bar{k}\left[-y-2y\right] ds$$

$$= \int_{-a}^{a}\int_{0}^{b} -y-2y\; dx\, dy$$

$$= \int_{-a}^{a} \left[-y-2y\right]\left[x\right]_0^b \; dxy.$$

$$= b\left[-3\frac{y^2}{2}\right]_{-a}^{a} \quad d$$

$$-3b\left[\frac{a^2}{2} + \frac{a^2}{2}\right.$$

$$-3.$$

$$\int_{-a}^{a}\int_{0}^{b} -3y\; dy\, dx$$

$$\left[-3\,\frac{y^2}{2}\right]_0^b$$

$$\int_{-a}^{a} \frac{-3}{2}\, b^2\; dx$$

$$\frac{-3}{2}\, b^2 \left[x\right]_{-a}^{a}$$

$$= -3ab^2$$